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The leading root of the partial theta function

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Abstract

I study the leading root $x_0(y)$ of the partial theta function $\Theta_0(x, y) = \sum_{n=0}^{\infty} x^n y^{n(n-1)/2}$, considered as a formal power series. I prove that all the coefficients of $-x_0(y)$ are strictly positive. Indeed, I prove the stronger results that all the coefficients of $-1/x_0(y)$ after the constant term 1 are strictly negative, and all the coefficients of $1/x_0(y)^2$ after the constant term 1 are strictly negative except for the vanishing coefficient of y^3.

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1. Introduction

Consider a formal power series of the form

$$f(x, y) = \sum_{n=0}^{\infty} \alpha_n x^n y^{n(n-1)/2} \tag{1.1}$$

where the coefficients $(\alpha_n)_{n=0}^{\infty}$ belong to a commutative ring-with-identity-element R and we impose the normalization $\alpha_0 = \alpha_1 = 1$. We can regard f as a formal power series in y whose

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coefficients are polynomials in x, i.e. $f \in R[x][[y]]$. Then, for any formal power series $X(y)$ with coefficients in R, the composition $f(X(y), y)$ makes sense as a formal power series in y. In particular, it is easy to see — either by the implicit function theorem for formal power series [18, p. A.IV.37], [44, Proposition 3.1] or by a direct inductive argument — that there exists a unique formal power series $x_0(y) \in R[[y]]$ satisfying $f(x_0(y), y) = 0$, which I call the "leading root" of f. Since $x_0(y)$ obviously has constant term -1, it is convenient to write $x_0(y) = -\xi_0(y)$ where $\xi_0(y) = 1 + O(y)$.

Among the interesting series $f(x, y)$ of this type are the "partial theta function" [9, Chapter 13], [10, Chapter 6]

$$\Theta_0(x, y) = \sum_{n=0}^{\infty} x^n y^{n(n-1)/2} \tag{1.2}$$

and the "deformed exponential function" [34,33,32,45–48]

$$F(x, y) = \sum_{n=0}^{\infty} \frac{x^n}{n!} y^{n(n-1)/2}. \tag{1.3}$$

More generally one can consider the rescaled three-variable Rogers–Ramanujan function [47]

$$\widetilde{R}(x, y, q) = \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(1+q)(1+q+q^2)\ldots(1+q+\cdots+q^{n-1})}, \tag{1.4}$$

which reduces to the foregoing when $q = 0$ and $q = 1$, respectively.

I have recently discovered empirically that the power series $\xi_0(y)$ has all nonnegative (in fact strictly positive) coefficients in the first two cases, and more generally in the third case whenever $q > -1$. More precisely, I have verified this for Θ_0 and F through orders y^{6999} and y^{899}, respectively, using a formula [47] that relates $\xi_0(y)$ to the series expansion of $\log f(x, y)$. For $\widetilde{R}$, I have proven [47] that $\xi_0(y, q)$ has the form

$$\xi_0(y, q) = 1 + \sum_{n=1}^{\infty} \frac{P_n(q)}{Q_n(q)} y^n \tag{1.5}$$

where

$$Q_n(q) = \prod_{k=2}^{\infty} \left(1 + q + \cdots + q^{k-1}\right)^{\lfloor n/\binom{k}{2} \rfloor} \tag{1.6}$$

and $P_n(q)$ is a self-inversive polynomial in q with integer coefficients; and I have verified for $n \leqslant 349$ that $P_n(q)$ has *two* interesting positivity properties:

(a) $P_n(q)$ has all nonnegative coefficients. Indeed, all the coefficients are strictly positive except $[q^1]P_5(q) = 0$.
(b) $P_n(q) > 0$ for $q > -1$.

Of course, I conjecture that these properties hold for all n, but I have (as yet) no proof.

The main purpose of this paper is to give a simple proof of the coefficientwise positivity of $\xi_0(y)$ in the case of the partial theta function (1.2):

Theorem 1.1. *For the partial theta function* (1.2)*, the formal power series*

$$\xi_0(y) = 1 + y + 2y^2 + 4y^3 + 9y^4 + 21y^5 + 52y^6 + 133y^7 + 351y^8 + 948y^9 + 2610y^{10} + \cdots \tag{1.7}$$

has strictly positive coefficients.

In fact, with a bit more work one can prove a pair of successively stronger results:

Theorem 1.2. *For the partial theta function* (1.2)*, the formal power series*

$$1/\xi_0(y) = 1 - y - y^2 - y^3 - 2y^4 - 4y^5 - 10y^6 - 25y^7 - 66y^8 - 178y^9 - 490y^{10} - \cdots \tag{1.8}$$

has strictly negative coefficients after the constant term 1.

Theorem 1.3. *For the partial theta function* (1.2)*, the formal power series*

$$1/\xi_0(y)^2 = 1 - 2y - y^2 \qquad - y^4 - 2y^5 - 7y^6 - 18y^7 - 50y^8 - 138y^9 - 386y^{10} - \cdots \tag{1.9}$$

has strictly negative coefficients after the constant term 1 *except for the vanishing coefficient of* y^3.

For further discussion of the relationship between these results, see Section 7.

In addition, I have discovered empirically a vast strengthening of Theorems 1.1 and 1.2. Please note first that any power series $g(y) = 1 + \sum_{n=1}^{\infty} a_n y^n \in \mathbb{Z}[\![y]\!]$ can be written uniquely as an infinite product $g(y) = \prod_{m=1}^{\infty}(1 - y^m)^{-c_m}$ with coefficients $c_m \in \mathbb{Z}$.[2] We then have:

Conjecture 1.4. *For the partial theta function* (1.2)*, when the formal power series* $\xi_0(y)$ *is written in the form* $\xi_0(y) = \prod_{m=1}^{\infty}(1 - y^m)^{-c_m}$*, the coefficient sequence*

$$(c_m)_{m=1}^{\infty} = 1, 1, 2, 4, 10, 23, 61, 157, 426, 1163, 3253, 9172, 26\,236, 75\,634, \ldots \tag{1.10}$$

is strictly positive ($c_m > 0$)*, increasing* ($\Delta c \geqslant 0$)*, strictly convex* ($\Delta^2 c > 0$)*, and satisfies* $\Delta^k c \geqslant 0$ *for* $k = 3, 4$. [*By contrast, the sequence* $\Delta^5 c$ *starts with* -3.]

[2] See e.g. [6, Theorem 10.3]. Some authors [16,35], [43, pp. 20–21] call $(a_n)_{n=1}^{\infty}$ the *Euler transform* of $(c_m)_{m=1}^{\infty}$, and $(c_m)_{m=1}^{\infty}$ the *inverse Euler transform* of $(a_n)_{n=1}^{\infty}$. However, this should not be confused with an unrelated (and more widely used) "Euler transformation" of sequences, involving binomial coefficients.

Conjecture 1.5. *For the partial theta function* (1.2)*, when the formal power series* $\xi_0(y)$ *is written in the form* $2 - 1/\xi_0(y) = \prod_{m=1}^{\infty}(1-y^m)^{-c'_m}$*, the coefficient sequence*

$$\left(c'_m\right)_{m=1}^{\infty} = 1, 0, 0, 1, 2, 6, 15, 40, 110, 303, 853, 2419, 6950, 20\,110, \ldots \tag{1.11}$$

is nonnegative and convex.

I have verified these conjectures through order y^{6999}, but I have no idea how to prove them. Perhaps one should try to find a combinatorial interpretation of the coefficients (c_m) and (c'_m).

The series $\xi_0(y)$ appears to possess one further striking property, which I have again verified through order y^{6999}:

Conjecture 1.6. *For the partial theta function* (1.2)*, the coefficient sequence of* $\xi_0(y) = \sum_{n=0}^{\infty} a_n y^n$ *is log convex, i.e.* $a_{n-1}a_{n+1} \geqslant a_n^2$ *for all* $n \geqslant 1$.

A classic theorem of Kaluza [27] relates Conjecture 1.6 to Theorem 1.2: namely, if the coefficient sequence $(a_n)_{n=0}^{\infty}$ of a formal power series f is strictly positive and log convex, then $1/f$ has nonpositive coefficients after the constant term; and if in addition $a_0a_2 > a_1^2$, then $1/f$ has strictly negative coefficients after the constant term.[3] But it is easily seen that the converse does not hold.[4] So Conjecture 1.6, if true, is a strengthening of Theorem 1.2.

The plan of this paper is as follows: I begin (Section 2) by recalling two identities for the partial theta function, which will play a central role in the proofs of Theorems 1.1–1.3. I then give, in successive sections, the proofs of Theorems 1.1–1.3 (Sections 3–5). Next I state and prove some identities for the three-variable Rogers–Ramanujan function (1.4) that may turn out to be useful in proving the conjectures concerning its leading root (Section 6). Finally, I place Theorems 1.1–1.3 in a more general context [42] and mention some stronger properties possessed by the power series $\xi_0(y)$ for the cases (1.2)–(1.4) that appear empirically to be true (Section 7).

A MATHEMATICA file `partialtheta_xi0.m` containing the series $\xi_0(y)$ for the partial theta function through order y^{6999} is available as an ancillary file with the preprint version of this paper at arXiv.org.

2. Identities for the partial theta function

In this section we recall a pair of identities for the partial theta function (1.2) that will serve as the foundation for our proofs of Theorems 1.1–1.3. We use the standard notation $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$ and $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$.

Lemma 2.1. *The partial theta function* (1.2) *satisfies*

$$\Theta_0(x,y) = (y;y)_\infty(-x;y)_\infty \sum_{n=0}^{\infty} \frac{y^n}{(y;y)_n(-x;y)_n}, \tag{2.1}$$

[3] The assertion about strict negativity is not explicitly stated by Kaluza [27], but it follows easily from his proof. See also [30, Lemma 2.2].

[4] For instance, let $f(y) = 1/(1-y-cy^2)$; then $1/f$ has nonpositive coefficients after the constant term whenever $c \geqslant 0$; but the coefficients of f are log convex only when $c = 0$.

$$\Theta_0(x,y) = (-x;y)_\infty \sum_{n=0}^{\infty} \frac{(-x)^n y^{n^2}}{(y;y)_n(-x;y)_n} \tag{2.2}$$

as formal power series and as analytic functions on $(x,y) \in \mathbb{C} \times \mathbb{D}$.[5]

In order to make this paper self-contained for readers who (like myself!) are not experts in q-series, we provide here an easy proof of (2.1) that uses nothing more than Euler's first and second identities [22, Eqs. (1.3.15) and (1.3.16)]

$$\frac{1}{(t;q)_\infty} = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n}, \tag{2.3}$$

$$(t;q)_\infty = \sum_{n=0}^{\infty} \frac{(-t)^n q^{n(n-1)/2}}{(q;q)_n} \tag{2.4}$$

valid for $(t,q) \in \mathbb{D} \times \mathbb{D}$ and $(t,q) \in \mathbb{C} \times \mathbb{D}$, respectively.

Proof of (2.1) [19,2]. Write

$$\Theta_0(x,y) = \sum_{n=0}^{\infty} x^n y^{n(n-1)/2} \frac{(y;y)_\infty}{(y;y)_n (y^{n+1};y)_\infty} \tag{2.5}$$

and insert Euler's first identity for $1/(y^{n+1};y)_\infty$: we obtain

$$\Theta_0(x,y) = (y;y)_\infty \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(y;y)_n} \sum_{k=0}^{\infty} \frac{y^{(n+1)k}}{(y;y)_k} \tag{2.6a}$$

$$= (y;y)_\infty \sum_{k=0}^{\infty} \frac{y^k}{(y;y)_k} \sum_{n=0}^{\infty} \frac{(xy^k)^n y^{n(n-1)/2}}{(y;y)_n} \tag{2.6b}$$

$$= (y;y)_\infty \sum_{k=0}^{\infty} \frac{y^k}{(y;y)_k} \bigl(-xy^k;y\bigr)_\infty \quad \text{by Euler's second identity} \tag{2.6c}$$

$$= (y;y)_\infty (-x;y)_\infty \sum_{k=0}^{\infty} \frac{y^k}{(y;y)_k(-x;y)_k}. \qquad \square \tag{2.6d}$$

And here is an easy proof of both (2.1) and (2.2) that uses only Heine's first and second transformations [22, Eqs. (1.4.1) and (1.4.5)]

[5] Here $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$. The right-hand sides of (2.1) and (2.2) have removable singularities at $x = -y^{-k}$ $(k = 0,1,2,\ldots)$. To see that these singularities are indeed removable, just rewrite $(-x;y)_\infty/(-x;y)_n$ as $(-xy^n;y)_\infty$.

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty}\, {}_2\phi_1(c/b, z; az; q, b), \tag{2.7} $$

$$ {}_2\phi_1(a,b;c;q,z) = \frac{(c/a;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty}\, {}_2\phi_1(abz/c, a; az; q, c/a) \tag{2.8} $$

for the basic hypergeometric function

$$ {}_2\phi_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n. \tag{2.9} $$

Here (2.7) is valid when $|q| < 1$, $|z| < 1$ and $0 < |b| < 1$, while (2.8) is valid when $|q| < 1$, $|z| < 1$ and $0 < |c| < |a|$.

Proof of (2.1) and (2.2) [11]. In (2.7) and (2.8), set $b = q$ and $z = -x/a$, then take $a \to \infty$ and $c \to 0$; we obtain (2.1) and (2.2) with y renamed as q. $\square$

Remarks. Identity (2.1) goes back to Heine in 1847 [24, bottom p. 306], who derived it (as here) as a limiting case of his fundamental transformation (2.7).[6] In the modern literature it can be found in Fine [21, Eq. (7.32)].

I don't know who first found identity (2.2); I would be grateful to any reader who can supply a reference. I first learned (2.2) from the paper of Andrews and Warnaar [11, Eq. (2.1)], but it is surely much older.

The elementary proof of (2.1) given here is in essence that given recently by Chen and Xia [19, Eq. (2.10)] and Alladi [2, second proof of (1.6)].[7] Our proof of (2.1) and (2.2) using Heine's transformations follows Andrews and Warnaar [11, Eq. (2.1)],[8] but at least for (2.1) the argument goes back to Heine himself [24, p. 306]. Note also that if one takes this latter proof of (2.1) and inserts in it the standard *proof* of Heine's first transformation [22, Section 1.4], one obtains the elementary proof of (2.1).

A combinatorial proof of (2.1) was given recently by Yee [53, Theorem 2.1], and combinatorial proofs of both (2.1) and the equality (2.1) = (2.2) were given recently by Kim [29, Section 2].

Many generalizations of (2.1)/(2.2), with additional parameters, are known. For instance, (2.1)/(2.2) can be extended from the partial theta function to more general basic hypergeometric functions ${}_1\phi_1$.[9] Another generalization of (2.1) appears in Ramanujan's lost notebook [40, p. 40], [10, Entry 6.3.1]; it was proven by Andrews [4, Section 4] and recently re-proven combinatorially by Kim [29, Section 4]. An even more general formula was proven subsequently by Andrews [5, Section 3], with a later simplification and further generalization by R.P. Agarwal [1]; see also

6 Heine makes the change of variables $x = -zq$ and $y = q^2$. The formula in [24, bottom p. 306] has a typographical error in which the factor y^n $(= q^{2n})$ in the numerator of the right-hand side is inadvertently omitted. The correct formula can be found in the 1878 edition of Heine's book [25, p. 107].

7 Alladi's Eq. (1.6) is equivalent to our (2.1) under the substitutions $x = -aq$ and $y = q^2$.

8 See also Andrews [7, proof of Theorem 1] for this proof of (2.1).

9 For the case of (2.2), this generalization can be found in papers of Bhargava and Adiga [17] and Srivastava [49]. A special case of this generalization can be found in Ramanujan's second notebook [39, Entry 9 in Chapter 16], [14, p. 18] and again in a page published with the lost notebook [40, p. 362], [10, Entry 1.6.1]. A combinatorial proof of this special case was recently given by Berndt, Kim and Yee [15, Theorem 5.6].

[10, Sections 6.2 and 6.3]. A formula generalizing the equality (2.1) = (2.2) appears in Ramanujan's lost notebook [40, p. 40], [10, Entry 1.6.7] and has an easy q-series proof [10, p. 27]; a combinatorial proof was recently given by Kim [29, Section 4].

A very beautiful formula for the sum of two partial theta functions, which generalizes both (2.1) and the Jacobi triple product identity, was found by Warnaar [51, Theorem 1.5]. A closely related identity for the product of two partial theta functions, which also generalizes (2.1), was found by Andrews and Warnaar [11, Theorem 1.1] and recently re-proven combinatorially by Kim [29, Section 3]; see also [10, Section 6.6].

Finally, Andrews [8, Theorem 5] has recently proven a finite-sum generalization of (2.1):

$$\sum_{n=0}^{N} \frac{x^n y^{n(n-1)/2}}{(y;y)_{N-n}} = (-x;y)_N \sum_{n=0}^{N} \frac{y^n}{(y;y)_n(-x;y)_n}. \tag{2.10}$$

Likewise, by using [8, Corollary 3] with $\alpha = q$, $\tau = -x/\beta$ and taking $\beta \to \infty$ and $\gamma \to 0$, one can derive a finite-sum generalization of (2.2):

$$\sum_{n=0}^{N} \frac{x^n y^{n(n-1)/2}}{(y;y)_{N-n}} = (-x;y)_N \sum_{n=0}^{N} \frac{(-x)^n y^{n^2}}{(y;y)_n(-x;y)_n(y;y)_{N-n}}. \tag{2.11}$$

See also [41] for a combinatorial proof of the finite Heine transformation that underlies (2.10) and (2.11).

3. Proof of Theorem 1.1

The proof of Theorem 1.1 can be based on either (2.1) or (2.2). For concreteness let us use (2.1), which we rewrite as

$$\Theta_0(x,y) = (y;y)_\infty(-xy;y)_\infty\left[1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n(-xy;y)_{n-1}}\right]. \tag{3.1}$$

So $\Theta_0(-\xi_0(y), y) = 0$ is equivalent to

$$\xi_0(y) = 1 + \sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n(y\xi_0(y);y)_{n-1}} \tag{3.2a}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{y^n}{\prod_{j=1}^{n}(1-y^j)\prod_{j=1}^{n-1}[1-y^j\xi_0(y)]}. \tag{3.2b}$$

This formula can be used iteratively to determine $\xi_0(y)$, and in particular to prove the strict positivity of its coefficients:

Proposition 3.1. *Define the map $\mathcal{F} : \mathbb{Z}[\![y]\!] \to \mathbb{Z}[\![y]\!]$ by*

$$(\mathcal{F}\xi)(y) = 1 + \sum_{n=1}^{\infty} \frac{y^n}{\prod_{j=1}^{n}(1-y^j)\prod_{j=1}^{n-1}[1-y^j\xi(y)]}, \tag{3.3}$$

and define a sequence $\xi_0^{(0)}, \xi_0^{(1)}, \ldots \in \mathbb{Z}[\![y]\!]$ *by* $\xi_0^{(0)} = 1$ *and* $\xi_0^{(k+1)} = \mathcal{F}\xi_0^{(k)}$. *Then*

$$\xi_0^{(0)} \preccurlyeq \xi_0^{(1)} \preccurlyeq \xi_0^{(2)} \preccurlyeq \cdots \preccurlyeq \xi_0 \tag{3.4}$$

(*where* $f \preccurlyeq g$ *denotes* $[y^n]f(y) \leqslant [y^n]g(y)$ *for all* n) *and*

$$\xi_0^{(k)}(y) = \xi_0(y) + O\big(y^{3k+1}\big). \tag{3.5}$$

In particular, $\lim_{k\to\infty} \xi_0^{(k)}(y) = \xi_0(y)$ *in the sense of convergence of formal power series* (*i.e. every coefficient eventually stabilizes at its limit*), *and* $\xi_0(y)$ *has strictly positive coefficients.*

Proof. If $f(y)$ and $g(y)$ are formal power series satisfying $0 \preccurlyeq f \preccurlyeq g$, then it is easy to see that $\prod_{j=1}^{n-1}[1 - y^j f(y)]^{-1} \preccurlyeq \prod_{j=1}^{n-1}[1 - y^j g(y)]^{-1}$ and hence $0 \preccurlyeq \mathcal{F}f \preccurlyeq \mathcal{F}g$. Applying this repeatedly to the obvious inequality $0 \preccurlyeq \xi_0^{(0)} \preccurlyeq \xi_0^{(1)}$, we obtain $\xi_0^{(0)} \preccurlyeq \xi_0^{(1)} \preccurlyeq \xi_0^{(2)} \preccurlyeq \cdots$.

Likewise, if $f(y)$ and $g(y)$ are formal power series satisfying $f(y) - g(y) = O(y^\ell)$ for some $\ell \geqslant 0$, then it is not hard to see that $(\mathcal{F}f)(y) - (\mathcal{F}g)(y) = O(y^{\ell+3})$ [coming from the $n = 2$ term in (3.3) and the $j = 1$ factor in the second product]. Applying this repeatedly to the obvious fact $\xi_0^{(1)}(y) - \xi_0^{(0)}(y) = O(y)$, we obtain $\xi_0^{(k+1)}(y) - \xi_0^{(k)}(y) = O(y^{3k+1})$. It follows that $\xi_0^{(k)}(y)$ converges as $k \to \infty$ (in the topology of formal power series) to a limiting series $\xi_0^{(\infty)}(y)$, and that this limiting series satisfies $\mathcal{F}\xi_0^{(\infty)} = \xi_0^{(\infty)}$. But this means, by (3.1)/(3.2b), that $\xi_0^{(\infty)}(y) = \xi_0(y)$. It also follows that $\xi_0^{(k)}(y) = \xi_0(y) + O(y^{3k+1})$.

Since $\xi_0^{(1)}(y)$ manifestly has strictly positive coefficients, it follows from (3.4) that $\xi_0(y)$ also has strictly positive coefficients. □

Remarks. 1. By a slightly more refined version of the same argument, one can prove inductively that

$$\xi_0^{(k+1)}(y) - \xi_0^{(k)}(y) = y^{3k+1} + (4k+2)y^{3k+2} + (4k+1)(2k+3)y^{3k+3} + O\big(y^{3k+4}\big) \tag{3.6}$$

for $k \geqslant 1$, and hence that

$$\xi_0(y) - \xi_0^{(k)}(y) = y^{3k+1} + (4k+2)y^{3k+2} + (4k+1)(2k+3)y^{3k+3} + O\big(y^{3k+4}\big) \tag{3.7}$$

for $k \geqslant 1$.

2. The series

$$\xi_0^{(1)}(y) = 1 + \sum_{n=1}^{\infty} \frac{y^n}{(y;y)_n (y;y)_{n-1}} = 1 + \frac{1 - \Theta_0(-y,y)}{(y;y)_\infty^2} \tag{3.8a}$$

$$= 1 + y + 2y^2 + 4y^3 + 8y^4 + 15y^5 + 27y^6 + 47y^7 + 79y^8 + \cdots \tag{3.8b}$$

enumerates weakly unimodal sequences of positive integers (also called "stacks" or "stack polyominoes") by total weight [12,52], [50, Section 2.5], [35, sequence A001523]. It would be interesting to seek combinatorial interpretations of $\xi_0^{(k)}(y)$ for $k \geqslant 2$, or at least of $\xi_0(y)$.[10]

3. Empirically I have observed that the $\xi_0^{(k)}$ obey inequalities stronger than (3.4), namely $\xi_0^{(k)}/\xi_0^{(k-1)} \succcurlyeq 1$ for $k \geqslant 1$. I have verified this through order y^{500} for $1 \leqslant k \leqslant 20$, but I do not see how to prove it. If true, this exhibits $\xi_0(y)$ as an infinite product of nonnegative series $\xi_0^{(k)}(y)/\xi_0^{(k-1)}(y)$, reminiscent of but different from Conjecture 1.4.

4. The recursion $\xi_0^{(k+1)} = \mathcal{F}\xi_0^{(k)}$ could alternatively have been started with $\xi_0^{(0)} = 0$ instead of $\xi_0^{(0)} = 1$. The only difference is that we would then have $\xi_0^{(k)}(y) - \xi_0(y) = O(y^{3k})$ instead of $O(y^{3k+1})$. In this case

$$\xi_0^{(1)}(y) = \sum_{n=0}^{\infty} \frac{y^n}{(y;y)_n} = \frac{1}{(y;y)_\infty} = \sum_{n=0}^{\infty} p(n) y^n \tag{3.9a}$$

$$= 1 + y + 2y^2 + 3y^3 + 5y^4 + 7y^5 + 11y^6 + 15y^7 + 22y^8 + \cdots \tag{3.9b}$$

is the generating function for all partitions of the integer n. Perhaps $\xi_0^{(k)}(y)$ for $k \geqslant 2$ have a simpler interpretation with this choice of $\xi_0^{(0)}$.[11]

Furthermore, with this choice of $\xi_0^{(0)}$ we have empirically not only $\xi_0^{(k)}/\xi_0^{(k-1)} \succcurlyeq 1$ for $k \geqslant 2$, but in fact $\xi_0^{(k)}(y)/\xi_0^{(k-1)}(y) = \prod_{m=1}^{\infty}(1-y^m)^{-c_m^{(k)}}$ with nonnegative coefficients $c_m^{(k)}$. I have verified this through order y^{500} for $2 \leqslant k \leqslant 20$. If true, this implies Conjecture 1.4.

5. If we use (2.2) instead of (2.1), then we are led to the recursion based on the map $\mathcal{G}: \mathbb{Z}[\![y]\!] \to \mathbb{Z}[\![y]\!]$ defined by

$$(\mathcal{G}\xi)(y) = 1 + \sum_{n=1}^{\infty} \frac{\xi(y)^n y^{n^2}}{\prod_{j=1}^{n}(1-y^j)\prod_{j=1}^{n-1}[1-y^j\xi(y)]}. \tag{3.10}$$

Using $\xi_0^{(0)} = 1$, we have for this map the slower convergence $\xi_0^{(k)}(y) - \xi_0(y) = O(y^k)$ [coming from the $\xi(y)^n$ factor in the numerator of the $n=1$ term in (3.10)]. In this case the series

$$\xi_0^{(1)}(y) = 1 + \sum_{n=1}^{\infty} \frac{y^{n^2}}{(y;y)_n (y;y)_{n-1}} = 1 + \frac{1-\Theta_0(-y,y)}{(y;y)_\infty} \tag{3.11a}$$

$$= 1 + y + y^2 + y^3 + 2y^4 + 3y^5 + 5y^6 + 7y^7 + 10y^8 + \cdots \tag{3.11b}$$

enumerates n-stacks with strictly receding walls [12,52], [35, sequence A001522]. Once again we have empirically $\xi_0^{(k)}/\xi_0^{(k-1)} \succcurlyeq 1$ for $k \geqslant 1$; I have verified this through order y^{2000} for

[10] **Note added:** Thomas Prellberg [37] has recently found a combinatorial interpretation of $\xi_0(y)$ and $\xi_0^{(k)}(y)$ in terms of rooted trees enriched by stack polyominoes, using results from [38] and [13, Chapter 3].

[11] **Note added:** Thomas Prellberg [37] has found a combinatorial interpretation of $\xi_0^{(k)}(y)$ also for this choice of $\xi_0^{(0)}$.

$1 \leqslant k \leqslant 20$. Furthermore, for this map taking $\xi_0^{(0)} = 0$ yields $\xi_0^{(1)} = 1$, so we obtain the *same* sequence (shifted by one) with both initial conditions.[12]

It is useful to abstract what we have done here (see [47] for details and extensions). Consider a formal power series (with coefficients in a commutative ring-with-identity-element R)

$$f(x, y) = \sum_{n=0}^{\infty} a_n(y) x^n \tag{3.12}$$

where

(a) $a_0(0) = a_1(0) = 1$;
(b) $a_n(0) = 0$ for $n \geqslant 2$; and
(c) $a_n(y) = O(y^{\nu_n})$ with $\lim_{n\to\infty} \nu_n = \infty$.

Then it is easy to see that there exists a unique formal power series $\xi_0(y)$ with coefficients in R satisfying $f(-\xi_0(y), y) = 0$, and it has constant term 1. Let us rearrange $f(-\xi_0(y), y) = 0$ as

$$\xi_0(y) = 1 + \sum_{n=0}^{\infty} (-1)^n \hat{a}_n(y) \xi_0(y)^n, \tag{3.13}$$

where $\hat{a}_n(y)$ is defined by

$$\hat{a}_n(y) = \begin{cases} a_n(y) - 1 & \text{for } n = 0, 1, \\ a_n(y) & \text{for } n \geqslant 2. \end{cases} \tag{3.14}$$

Now suppose that the ring R carries a partial order compatible with the ring structure (typically we will have $R = \mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}$) and that

$$(-1)^n \hat{a}_n(y) \succcurlyeq 0 \quad \text{for all } n \geqslant 0, \tag{3.15}$$

where $f(y) \succcurlyeq 0$ means that f has all nonnegative coefficients. Then the recursion argument used in Proposition 3.1, applied to (3.13), shows that $\xi_0(y) \succcurlyeq 1 + \sum_{n=0}^{\infty} (-1)^n \hat{a}_n(y)$. The case treated here was

$$f(x, y) = \frac{\Theta_0(x, y)}{(y; y)_\infty (-xy; y)_\infty} = 1 + x + \sum_{n=1}^{\infty} \frac{y^n}{(y; y)_n (-xy; y)_{n-1}}. \tag{3.16}$$

The value of the identity (2.1) or (2.2) for our purposes is that powers of x on the left-hand side are transformed into powers of $-x$ on the right-hand side, so that (3.15) holds for the latter.

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12 **Note added:** Thomas Prellberg [37] has found a combinatorial interpretation also for these $\xi_0^{(k)}(y)$.

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2 on the strict negativity of the coefficients of $\xi_0(y)^{-1}$ after the constant term 1. It is convenient to state and prove first an abstract result of this form [47]; then we verify the hypotheses of this abstract result in our specific case.

Proposition 4.1. *Consider a formal power series* (*with coefficients in a partially ordered commutative ring R*)

$$f(x,y)=\sum_{n=0}^{\infty} a_n(y)x^n \tag{4.1}$$

where

(a) $a_0(0)=a_1(0)=1$;
(b) $a_n(0)=0$ *for* $n\geqslant 2$; *and*
(c) $a_n(y)=O(y^{\nu_n})$ *with* $\lim_{n\to\infty}\nu_n=\infty$.

Let $\xi_0(y)$ be the unique power series satisfying $f(-\xi_0(y),y)=0$. Suppose that

$$1-\frac{a_1(y)}{a_0(y)}\succcurlyeq 0 \tag{4.2}$$

and that

$$(-1)^n\frac{a_n(y)}{a_0(y)}\succcurlyeq 0 \quad \textit{for all } n\geqslant 2. \tag{4.3}$$

Then

$$\xi_0(y)^{-1}\preccurlyeq \frac{a_1(y)}{a_0(y)}-\sum_{n=2}^{\infty}(-1)^n\frac{a_n(y)}{a_0(y)}\preccurlyeq 1. \tag{4.4}$$

Proof. Start from the equation $\sum_{n=0}^{\infty}(-1)^n a_n(y)\xi_0(y)^n=0$, divide by $a_0(y)\xi_0(y)$, and bring $\xi_0(y)^{-1}$ to the left-hand side: we have

$$\xi_0(y)^{-1}=\frac{a_1(y)}{a_0(y)}-\sum_{n=2}^{\infty}(-1)^n\frac{a_n(y)}{a_0(y)}\xi_0(y)^{n-1}. \tag{4.5}$$

Now write $\xi_0(y)^{-1}=1-\psi(y)$: we obtain

$$\psi(y)=1-\frac{a_1(y)}{a_0(y)}+\sum_{n=2}^{\infty}(-1)^n\frac{a_n(y)}{a_0(y)}\bigl[1-\psi(y)\bigr]^{-(n-1)}. \tag{4.6}$$

By hypothesis (4.6) is of the form

$$\psi(y) = b_1(y) + \sum_{n=2}^{\infty} b_n(y)\bigl[1 - \psi(y)\bigr]^{-(n-1)} \tag{4.7}$$

where $b_n(y) \succcurlyeq 0$ and $b_n(y) = O(y)$ for all $n \geqslant 1$. An iterative argument as in the proof of Proposition 3.1 then proves that $\psi(y) \succcurlyeq 0$ and in fact

$$\psi(y) \succcurlyeq 1 - \frac{a_1(y)}{a_0(y)} + \sum_{n=2}^{\infty} (-1)^n \frac{a_n(y)}{a_0(y)}. \qquad \square \tag{4.8}$$

Proof of Theorem 1.2. This time we find it convenient to use (2.2) instead of (2.1). We therefore apply Proposition 4.1 to the power series

$$f(x,y) = \frac{\Theta_0(x,y)}{(-xy;y)_\infty} = 1 + x + \sum_{n=1}^{\infty} \frac{(-x)^n y^{n^2}}{(y;y)_n (-xy;y)_{n-1}} \tag{4.9a}$$

$$= 1 + x - \frac{xy}{1-y} + \sum_{n=2}^{\infty} \frac{(-x)^n y^{n^2}}{(y;y)_n (-xy;y)_{n-1}}. \tag{4.9b}$$

The first three terms in (4.9b) give $a_0(y) = 1$ and $a_1(y) = 1 - y/(1-y)$, so that $1 - a_1(y)/a_0(y) = y/(1-y) \succcurlyeq 0$. On the other hand, the final sum in (4.9b) is manifestly a power series with nonnegative coefficients in $-x$ and y, which proves that $(-1)^m a_m(y) \succcurlyeq 0$ for all $m \geqslant 2$. $\square$

Remarks. 1. We can obtain an explicit formula for the coefficients $a_m(y)$ by inserting into (4.9b) the expansion [3, Theorem 3.3]

$$\frac{1}{(-xy;y)_{n-1}} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k-2 \\ k \end{bmatrix}_y (-xy)^k \quad \text{for } n \geqslant 2 \tag{4.10}$$

where the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} \quad \text{for } 0 \leqslant k \leqslant n. \tag{4.11}$$

This yields

$$f(x,y) = 1 + x - \frac{xy}{1-y} + \sum_{n=2}^{\infty} \sum_{k=0}^{\infty} \begin{bmatrix} n+k-2 \\ k \end{bmatrix}_y (-xy)^k \frac{(-x)^n y^{n^2}}{(y;y)_n}. \tag{4.12}$$

Extracting the coefficient of x^m for $m = n + k \geqslant 2$, we have

$$(-1)^m a_m(y) = y^m \sum_{n=2}^{m} \begin{bmatrix} m-2 \\ m-n \end{bmatrix}_y \frac{y^{n(n-1)}}{(y;y)_n}. \tag{4.13}$$

Since the q-binomial coefficients are polynomials in q with nonnegative integer coefficients [3, Theorem 3.2 or 3.6], we see once again that $(-1)^m a_m(y) \succcurlyeq 0$ for all $m \geqslant 2$. We also see from (4.13) that $a_m(y)$ is a rational function of the form $a_m(y) = P_m(y)/(y;y)_m$ where $P_m(y)$ is a polynomial with integer coefficients.

2. It would be interesting to seek a combinatorial interpretation of the coefficients of $1 - 1/\xi_0(y)$, analogously to what Prellberg [37] has done for $\xi_0(y)$ [see footnotes 10–12 above].

5. Proof of Theorem 1.3

Next we prove Theorem 1.3. It is convenient once again to state and prove first an abstract result [47], and then verify the hypotheses of this abstract result in our specific case.

Proposition 5.1. *Consider a formal power series $f(x,y)$ satisfying all the hypotheses of Proposition* 4.1. *Then*

$$\xi_0(y)^{-2} \preccurlyeq \left(\frac{a_1(y)}{a_0(y)}\right)^2 - 2\sum_{n=2}^{\infty}(-1)^n \frac{a_n(y)}{a_0(y)}\left(\frac{a_0(y)}{a_1(y)}\right)^{n-2}. \tag{5.1}$$

Proof. Divide both sides of (4.5) by $\xi_0(y)$ and then insert (4.5) in the first term on the right-hand side: we obtain

$$\xi_0(y)^{-2} = \frac{a_1(y)}{a_0(y)}\xi_0(y)^{-1} - \sum_{n=2}^{\infty}(-1)^n \frac{a_n(y)}{a_0(y)}\xi_0(y)^{n-2} \tag{5.2a}$$

$$= \left(\frac{a_1(y)}{a_0(y)}\right)^2 - \sum_{n=2}^{\infty}(-1)^n \frac{a_n(y)}{a_0(y)}\left[1 + \frac{a_1(y)}{a_0(y)}\xi_0(y)\right]\xi_0(y)^{n-2}. \tag{5.2b}$$

Now, by hypothesis we have $(-1)^n a_n(y)/a_0(y) \succcurlyeq 0$ for all $n \geqslant 2$. By Proposition 4.1 we have $\xi_0(y)^{-1} \preccurlyeq a_1(y)/a_0(y) \preccurlyeq 1$, hence $\xi_0(y)^{n-2} \succcurlyeq [a_0(y)/a_1(y)]^{n-2} \succcurlyeq 1$ for all $n \geqslant 2$. Finally, multiplying (4.5) by $\xi_0(y)$ and rearranging gives

$$\frac{a_1(y)}{a_0(y)}\xi_0(y) = 1 + \sum_{n=2}^{\infty}(-1)^n \frac{a_n(y)}{a_0(y)}\xi_0(y)^n \succcurlyeq 1. \tag{5.3}$$

Inserting these facts into (5.2) proves (5.1). □

Proof of Theorem 1.3. We again use (2.2) and thus apply Proposition 5.1 to the power series (4.9b). While proving Theorem 1.2 we showed that $a_0(y) = 1$, $a_1(y) = 1 - y/(1-y) \preccurlyeq 1$ and $(-1)^n a_n(y) \succcurlyeq 0$ for all $n \geqslant 2$, so all the hypotheses of Proposition 5.1 are satisfied. Furthermore, from either (4.9b) or (4.13) it is easy to see that

$$(-1)^n a_n(y) \succcurlyeq \frac{y^{n+2}}{(1-y)(1-y^2)} \succcurlyeq \frac{y^{n+2}}{1-y} \tag{5.4}$$

for all $n \geqslant 2$.[13] From (5.1) we then have

$$\xi_0(y)^{-2} \preccurlyeq \left(\frac{a_1(y)}{a_0(y)}\right)^2 - 2\sum_{n=2}^{\infty}(-1)^n \frac{a_n(y)}{a_0(y)}\left(\frac{a_0(y)}{a_1(y)}\right)^{n-2} \tag{5.5a}$$

$$\preccurlyeq \left(\frac{a_1(y)}{a_0(y)}\right)^2 - 2\sum_{n=2}^{\infty}(-1)^n \frac{a_n(y)}{a_0(y)} \tag{5.5b}$$

$$\preccurlyeq \left(\frac{1-2y}{1-y}\right)^2 - \frac{2y^4}{(1-y)^2} \tag{5.5c}$$

$$= 1 - 2y - y^2 - \sum_{n=4}^{\infty}(n-3)y^n, \tag{5.5d}$$

which proves Theorem 1.3. □

Remarks. 1. If we use (4.13) and expand the right-hand side of (5.1), we obtain

$$\xi_0(y)^{-2} \preccurlyeq 1 - 2y - y^2 \qquad - y^4 - 2y^5 - 7y^6 - 18y^7 - 49y^8 - 130y^9 - 343y^{10} - \cdots, \tag{5.6}$$

which differs from the exact $\xi_0(y)^{-2}$ starting at order y^8. The difference at order y^8 arises from a contribution to $\xi_0(y)$ that is proportional to $a_2(y)^2$. The full structure of the contributions to $\xi_0(y)$ and its powers can be read off the explicit implicit function formula [44]: see [47] for details.

2. It would be interesting to seek a combinatorial interpretation of the coefficients of $1 - 1/\xi_0(y)^2$, analogously to what Prellberg [37] has done for $\xi_0(y)$ [see footnotes 10–12 above].

6. Identities for $R(x, y, q)$

In this section we obtain some simple identities for the three-variable Rogers–Ramanujan function [47]

$$R(x, y, q) = \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(q;q)_n}. \tag{6.1}$$

The basic principle is in fact more general, and applies to an arbitrary power series of the form

$$F(x, q) = \sum_{n=0}^{\infty} \frac{a_n x^n}{(\alpha;q)_n}. \tag{6.2}$$

Lemma 6.1. *For arbitrary coefficients $(a_n)_{n=0}^{\infty}$ and an arbitrary constant α, we have*

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{(\alpha;q)_n} = \frac{1}{(\alpha;q)_\infty} \sum_{\ell=0}^{\infty} \frac{(-\alpha)^\ell q^{\ell(\ell-1)/2}}{(q;q)_\ell} \sum_{n=0}^{\infty} a_n \left(q^\ell x\right)^n \tag{6.3}$$

as formal power series.

[13] It suffices to take the term $n = 2$ in (4.9b) or (4.13), using the fact that all other terms are $\succcurlyeq 0$.

Proof. Write

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{(\alpha;q)_n} = \sum_{n=0}^{\infty} a_n x^n \frac{(\alpha q^n;q)_\infty}{(\alpha;q)_\infty} \tag{6.4}$$

and substitute Euler's second identity (2.4) for $(\alpha q^n;q)_\infty$, yielding

$$\sum_{n=0}^{\infty} \frac{a_n x^n}{(\alpha;q)_n} = \frac{1}{(\alpha;q)_\infty} \sum_{n=0}^{\infty} a_n x^n \sum_{\ell=0}^{\infty} \frac{(-\alpha q^n)^\ell q^{\ell(\ell-1)/2}}{(q;q)_\ell} \tag{6.5a}$$

$$= \frac{1}{(\alpha;q)_\infty} \sum_{\ell=0}^{\infty} \frac{(-\alpha)^\ell q^{\ell(\ell-1)/2}}{(q;q)_\ell} \sum_{n=0}^{\infty} a_n \bigl(q^\ell x\bigr)^n. \qquad \square \tag{6.5b}$$

Specializing to $a_n = y^{n(n-1)/2}$ and $\alpha = q$, we obtain a simple identity that expresses $R(x,y,q)$ in terms of the partial theta function:

Corollary 6.2. *The three-variable Rogers–Ramanujan function* (6.1) *satisfies*

$$R(x,y,q) = \frac{1}{(q;q)_\infty} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell q^{\ell(\ell+1)/2}}{(q;q)_\ell} \Theta_0\bigl(xq^\ell, y\bigr) \tag{6.6}$$

as formal power series and as analytic functions on $(x,y,q) \in \mathbb{C} \times \mathbb{D} \times \mathbb{D}$.

From Corollary 6.2 we can obtain a pair of identities for $R(x,y,q)$ that generalize (2.1)/(2.2) and reduce to them when $q = 0$:

Corollary 6.3. *We have*

$$R(x,y,q) = \frac{(y;y)_\infty}{(q;q)_\infty} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell q^{\ell(\ell+1)/2}}{(q;q)_\ell} \bigl(-xq^\ell;y\bigr)_\infty \sum_{n=0}^{\infty} \frac{y^n}{(y;y)_n \bigl(-xq^\ell;y\bigr)_n}, \tag{6.7}$$

$$R(x,y,q) = \frac{1}{(q;q)_\infty} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell q^{\ell(\ell+1)/2}}{(q;q)_\ell} \bigl(-xq^\ell;y\bigr)_\infty \sum_{n=0}^{\infty} \frac{\bigl(-xq^\ell\bigr)^n y^{n^2}}{(y;y)_n \bigl(-xq^\ell;y\bigr)_n} \tag{6.8}$$

as formal power series and as analytic functions on $(x,y,q) \in \mathbb{C} \times \mathbb{D} \times \mathbb{D}$.

Proof. Just substitute (2.1)/(2.2) into (6.6). $\square$

The function $\widetilde{R}$ defined in (1.4) is simply the rescaled version of R normalized to have $\alpha_0 = \alpha_1 = 1$:

$$\widetilde{R}(x,y,q) = R\bigl((1-q)x, y, q\bigr) = \sum_{n=0}^{\infty} \frac{x^n y^{n(n-1)/2}}{(1+q)(1+q+q^2)\dots(1+q+\dots+q^{n-1})}. \tag{6.9}$$

Unfortunately, I do not see how to imitate the proof of Theorem 1.1/Proposition 3.1 when $-1 < q < 0$ or $0 < q \leqslant 1$. But perhaps I am missing something.

7. Discussion

The positivity results stated in Theorems 1.1–1.3 can be better understood by placing them in the following general context [42]: For $\alpha \in \mathbb{R} \setminus \{0\}$, let us define the class $\mathcal{S}_\alpha$ to consist of those formal power series $f(y)$ with real coefficients and constant term 1 for which the series

$$\frac{f(y)^\alpha - 1}{\alpha} = \sum_{m=1}^{\infty} b_m(\alpha) y^m \tag{7.1}$$

has all nonnegative coefficients. The class $\mathcal{S}_0$ consists of those f for which the formal power series

$$\log f(y) = \sum_{m=1}^{\infty} b_m(0) y^m \tag{7.2}$$

has all nonnegative coefficients. The containment relations between the classes $\mathcal{S}_\alpha$ are given by the following fairly easy result [42]:

Proposition 7.1. *Let $\alpha, \beta \in \mathbb{R}$. Then $\mathcal{S}_\alpha \subseteq \mathcal{S}_\beta$ if and only if either*

(a) $\alpha \leqslant 0$ *and* $\beta \geqslant \alpha$, *or*
(b) $\alpha > 0$ *and* $\beta \in \{\alpha, 2\alpha, 3\alpha, \ldots\}$.

Moreover, the containment is strict whenever $\alpha \neq \beta$.

For the partial theta function (1.2), Theorem 1.1 states that $\xi_0 \in \mathcal{S}_1$; Theorem 1.2 states the stronger result that $\xi_0 \in \mathcal{S}_{-1}$ (and hence that $\xi_0 \in \mathcal{S}_\alpha$ for all $\alpha \geqslant -1$); and Theorem 1.3 states the yet stronger result that $\xi_0 \in \mathcal{S}_{-2}$ (and hence that $\xi_0 \in \mathcal{S}_\alpha$ for all $\alpha \geqslant -2$). This is best possible, since from

$$\frac{\xi_0(y)^\alpha - 1}{\alpha} = y + \frac{\alpha+3}{2} y^2 + \frac{(\alpha+2)(\alpha+7)}{6} y^3 + O\big(y^4\big) \tag{7.3}$$

we see immediately that $\xi_0 \notin \mathcal{S}_\alpha$ for $\alpha < -2$.

For the deformed exponential function (1.3), I conjecture that $\xi_0 \in \mathcal{S}_{-1}$ (see also [44, Example 4.3]), and I have verified this through order y^{899}. It follows from the asymptotics of $\xi_0(y)$ as $y \uparrow 1$ [45] that $\xi_0 \notin \mathcal{S}_\alpha$ for $\alpha < -1$.

For the function $\widetilde{R}$ defined in (1.4), I conjecture that $\xi_0 \in \mathcal{S}_{-1}$ for all $q > -1$, and I have verified this through order y^{349}. More strongly, I conjecture that for $q > -1$ there is a function $\alpha_\star(q)$ such that $\xi_0(y; q) \in \mathcal{S}_\alpha$ if and only if $\alpha \geqslant \alpha_\star(q)$, and having the following properties:

(a) $\alpha_\star(q) = -3$ for $-1 < q \leqslant -1/2$.
(b) $\alpha_\star(q)$ is strictly increasing on $-1/2 \leqslant q \leqslant 1$ and strictly decreasing on $q \geqslant 1$.
(c) $\alpha_\star(0) = -2$.
(d) $\alpha_\star(1) = -1$.
(e) $\alpha_\star(q) = \alpha_\star(1/q)$ for $q > 0$.

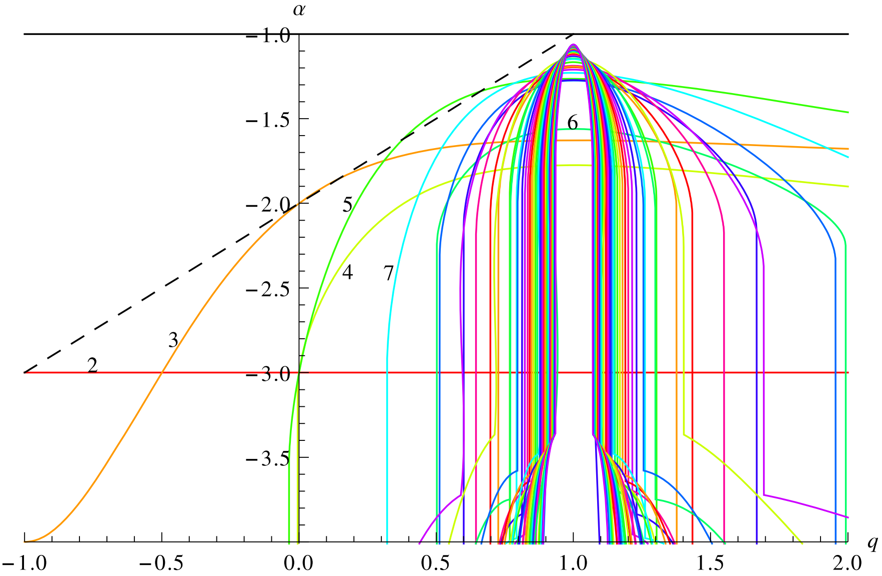

Fig. 1. Largest real root of $b_m(\alpha)$ as a function of q for $2 \leqslant m \leqslant 50$. The curves corresponding to $m \leqslant 7$ are labeled. The dashed black line is $\alpha = -2 + q$.

Since

$$\frac{\xi_0(y,q)^\alpha - 1}{\alpha} = \frac{y}{1+q} + \frac{\alpha+3}{2}\frac{y^2}{(1+q)^2} + O\big(y^3\big), \tag{7.4}$$

we see immediately that $\xi_0 \notin \mathcal{S}_\alpha$ for $\alpha < -3$. Fig. 1 shows numerical computations of the largest real root of $b_m(\alpha)$ [cf. (7.1)], as a function of $q \in (-1, 2]$, for $2 \leqslant m \leqslant 50$. The upper envelope of these curves should be $\alpha_\star(q)$. The simple conjecture $\alpha_\star(q) \leqslant -2 + q$ (shown as a dashed black line) barely fails in the range $0 < q \lesssim 0.145103$ because of the coefficient of y^3, and in the range $0.378619 \lesssim q \lesssim 0.660551$ because of the coefficient of y^5; but it appears to hold for $-1 < q \leqslant 0$. Indeed, for $-1 < q \leqslant 0$ it appears that $b_m(\alpha) \geqslant 0$ whenever $\alpha \geqslant -3$ and $m \neq 3$.

Finally, though in this paper I have treated $\xi_0(y)$ as a formal power series, it is not difficult to show [45,48], using Rouché's theorem, that $\xi_0(y)$ is in fact convergent for $|y| < \delta_1 \approx 0.2247945929$, where δ_1 is the positive root of $\sum_{\ell=-1}^{\infty} \delta^{\ell^2/2} = 2$. (This proof applies to both Θ_0 and F, and more generally to $\widetilde{R}$ for all $q \geqslant 0$.) Then the coefficientwise positivity established in Theorem 1.1 implies, by Pringsheim's theorem, that the first singularity of $\xi_0(y)$ for the partial theta function lies on the positive real axis, namely at the point $y = y_{01}^\star$ where the leading root $x_0(y)$ collides with the next root $x_1(y)$: this is the solution of the system

$$\Theta_0(x,y) = 0 \quad \text{and} \quad \frac{\partial \Theta_0(x,y)}{\partial x} = 0 \tag{7.5}$$

and lies at $(x, y) = (x^\star_{01}, y^\star_{01}) \approx (-2.3203769443, 0.3092493386)$.[14] Similarly, for the deformed exponential function (1.3) it is known [34,33,32] that $\xi_0(y)$ is analytic in a complex neighborhood of the real interval $0 < y < 1$; therefore, if the coefficients are indeed nonnegative, Pringsheim's theorem implies the striking fact that $\xi_0(y)$ is analytic in the whole unit disc $|y| < 1$.

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